ENTIRE CURVES AVOIDING GIVEN SETS IN $\mathbb{C}^n$

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ABSTRACT. Let $F \subset \mathbb{C}^n$ be a proper closed subset of $\mathbb{C}^n$ and $A \subset \mathbb{C}^n \setminus F$ at most countable ($n \geq 2$). We give conditions on F and A, under which there exists a holomorphic immersion (or a proper holomorphic embedding) $\varphi : \mathbb{C} \to \mathbb{C}^n$ with $A \subset \varphi(\mathbb{C}) \subset \mathbb{C}^n \setminus F$.

Let $F \subset \mathbb{C}^n$ be a proper closed subset of $\mathbb{C}^n$ and $A \subset \mathbb{C}^n \setminus F$ at most countable ($n \geq 2$). The aim of this note is to discuss conditions for F and A, under which there exists a holomorphic immersion (or a proper holomorphic embedding) $\varphi : \mathbb{C} \to \mathbb{C}^n$ with $A \subset \varphi(\mathbb{C}) \subset \mathbb{C}^n \setminus F$. Our main tool for constructing such mappings is Arakelian's approximation theorem (cf. [4, 11]).

The first result is a generalization of the main part of Theorem 1 in [7]. More precisely, we prove the following result.

Proposition 1. *Let F be a proper convex closed set in $\mathbb{C}^n$, $n \geq 2$. Then the following statements are equivalent:*

(i) either F is a complex hyperplane or it does not contain any complex hyperplane;

(ii) for any integer $k \geq 1$ and any two sets $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset \mathbb{C}$ and $\{a_1, a_2, \dots, a_k\} \subset \mathbb{C}^n \setminus F$, there exists a proper holomorphic embedding $\varphi : \mathbb{C} \to \mathbb{C}^n$ such that $\varphi(\alpha_j) = a_j$, $1 \leq j \leq k$, and $\varphi(\mathbb{C}) \subset \mathbb{C}^n \setminus F$.

(iii) the same as (ii) but for $k = 2$.

The equivalence of (i) and (iii) follows from the proof of Theorem 1 in [7]. For the convenience of the reader we repeat here the main idea of the proof of $(iii) \Longrightarrow (i)$. Observe that condition (iii) implies that the Lempert function of the domain $D := \mathbb{C}^2 \setminus F$ is identically zero, i.e.

$$\begin{aligned}\widetilde{k}_D(z, w) := \inf\{\alpha \geq 0 :\\ \exists f \in \mathcal{O}(\Delta, D) : f(0) = z,\ f(\alpha) = w\} = 0, \quad z, w \in D,\end{aligned}$$

where Δ denotes the open unit disc in $\mathbb{C}$. In the case when condition (i) is not satisfied we may assume (after a biholomorphic mapping) that $F = A \times \mathbb{C}$, where the closed convex set A, properly contained in $\mathbb{C}$, contains at least two points. Applying standard properties of $\widetilde{k}$, we

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have $\widetilde{k}_D(z,w) = \widetilde{k}_{\mathbb{C}\setminus A}(z',w')$, where $(z,w) = ((z',z''),(w',w'')) \in D$. Since $\widetilde{k}_{\mathbb{C}\setminus A}$ is not identically zero we end with a contradiction.

Hence, we only have to prove the implication $(i) \Longrightarrow (ii)$.

Proof. For simplicity of notations we shall consider only the case $n=2$.

If F is a complex line, we may assume that $F = \{z_2 = 0\}$. Considering an automorphism of the form $(z_1,z_2) \to (z_1 e^{\gamma z_1 z_2}, z_2 e^{-\gamma z_1 z_2})$ for a suitable constant γ, we may also assume that the second coordinates of the given points are pairwise different. Then there exist two one variable polynomials P and Q such that the mapping $t \to (t + P(e^{Q(t)}), e^{Q(t)})$ has the required property.

Assume now that F does not contain any complex line. The idea below comes from that of Theorem 8.5 in [9].

First, we shall prove by induction that for any $j \le k$ there is an automorphism Φ_j such that the set $\mathrm{co}(\Phi_j(F))$ does not contain any complex line and it does not have a common point with the set

$$\mathrm{co}(G_j) \cup \{\Phi_j(a_{j+1}),\dots,\Phi_j(a_k)\},$$

where $G_j := \{\Phi_j(a_1),\dots,\Phi_j(a_j)\}$ ($\mathrm{co}(M)$ denotes the convex hull of a closed set M in $\mathbb{C}^n$). Doing the induction step, we may assume that $\Phi_j = \mathrm{Id}$. Then, since F is convex and does not contain any complex line, after an affine change of coordinates one has that (cf. [2, 7])

$$F \subset H := \{\mathrm{Re}(z_1) \le -1, \mathrm{Re}(z_2) \le -1\},$$

$$\mathrm{co}(G_j) \subset \{\mathrm{Re}(z_1) \ge 0\},\ a_{j+1} \in \{\mathrm{Re}(z_2) \ge 0\}.$$

In addition, we may assume that the set $A := \{a_1,\dots,a_k\}$ of the given points and the strip $\{-1 < \mathrm{Re}(z_2) < 0\}$ do not have a common point. By Arakelian's theorem (cf. [4]), for $\varepsilon := \min\{1, \mathrm{dist}(F,A)\}$ we may find an entire function f such that

$$|f(t) - a_{j+1,1}| < \frac{\varepsilon}{2} \text{ if } \mathrm{Re}(t) \le -1,\ |f(t)| < \frac{\varepsilon}{2} \text{ if } \mathrm{Re}(t) \ge 0$$

and, in addition, $f(a_{j+1,2}) = 0$ (here, $a_{j+1,k}$ denotes the k-th coordinate of the point a_{j+1}). Then it is easy to see that the automorphism $\Phi_{j+1}(z_1,z_2) := (z_1 + f(z_2), z_2)$ has the required properties.

So, let F be a convex set, which does not contain any complex line and $F \cap \mathrm{co}(A) = \emptyset$. Then we may assume that (cf. [2, 7]) $F \subset H$, $A \subset \{\mathrm{Re}(z_1) \ge 1, \mathrm{Re}(z_2) \ge 0\}$, and, in addition, that $\mathrm{Re}(\alpha_j) \ge 1,\ 1 \le j \le k$.

Note that there exists an entire function g such that $|g(t)| \le 1$ if $\mathrm{Re}(t) \le -1$ and $g(a_{j,2}) = \alpha_j - a_{j,1}$ (cf. [4, 10]; this can be proved also directly, applying a standard interpolation process and Arakelian's theorem many times). Then, applying the automorphism $(z_1,z_2) \to$

$(z_1 + g(z_2), z_2)$, we may assume that $a_{j,1} = \alpha_j$ and $F \subset \{\mathrm{Re}(z_1) \leq 0, \mathrm{Re}(z_2) \leq -1\}$. Finally, we find, as above, an entire function h such that $|h(t)| < 1$ on the set $\mathrm{Re}(t) \leq 0$ and $h(\alpha_j) = a_{j,2}$. Hence, the mapping $t \to (t, h(t))$ has the required properties (in the new coordinates). □

The end of the proof shows that we may also prescribe values of finitely many derivatives of φ at the points of the given planar set.

Open problem. Is it true for an F as in (ii) of Proposition 1, that for any discrete set of points in $\mathbb{C}^n \setminus F$ there exists a proper holomorphic embedding of $\mathbb{C}$ in $\mathbb{C}^n$ avoiding F and passing through any of these points?

It is known that for any discrete set of points in $\mathbb{C}^n$ there exists a proper holomorphic embedding of $\mathbb{C}$ in $\mathbb{C}^n$ passing through any of the points of this set (Proposition 2 in [6]; cf. also Theorem 1 in [10] for $n \geq 3$). We have not been able to modify the proofs of [6, 10] to get a positive answer for the above question in the general case. Nevertheless, the following result gives a positive answer to the open problem in the case when F is a complex hyperplane.

Proposition 2. *If F is a union of at most $n-1$ $\mathbb{C}$-linearly independent complex hyperplanes in $\mathbb{C}^n$, then for any discrete set of points in $\mathbb{C}^n \setminus F$ there exists a proper holomorphic embedding of of $\mathbb{C}$ into $\mathbb{C}^n$ avoiding F and passing through any of these points.*

The proof of Proposition 2 will be a modification of the one in the case when F is the empty set (see Proposition 2 in [6]).

The key point is the following

Lemma 3. *Let K be a polynomially convex compact set in $\mathbb{C}^n$, A a set of finitely many points in K, and H a union of at most $n-1$ linearly independent complex hyperplanes in $\mathbb{C}^n$. For every $p, q \in \mathbb{C}^n \setminus (K \cup H)$ and every $\varepsilon > 0$, there exists an automorphism φ of $\mathbb{C}^n$ such that $\varphi(z) = z$, $z \in H \cap A$, $\varphi(p) = q$, and $|\varphi(z) - z| \leq \varepsilon$, $z \in K$.*

In view of Lemma 3, Proposition 2 follows by repeating step by step the proof of Proposition 2 in [6]. Starting with an embedding α_0 whose graph avoids H, the desired embedding α is constructed as the limit of a sequence of embeddings α_j with $\alpha_j = \varphi_j \circ \alpha_{j-1}$ $(j \geq 1)$, where the φ_j are automorphisms chosen by Lemma 3. Note that the graph of α avoids H by the Hurwitz theorem.

Proof of Lemma 3. After a linear change of coordinates, we may assume that $H \subset \{z_1 \cdots z_n = 0\}$ and that all the coordinates of the points in

$B := A \cup \{q\} \setminus H$ are non-zero. Applying an overshear of the form

$$w_1 = z_1 \exp(f(z_2, \dots, z_n)), w_2 = z_2, \dots, w_n = z_n,$$

where

$$f(z_2, \dots, z_n) := z_2 \dots z_n(\varepsilon + \sum_{j=2}^{n} \varepsilon^j z_j)$$

and ε is small enough, provides pairwise different products of the first $n-1$ coordinates of the points in B. Repeating this argument, we may assume the same for every $n-1$ coordinates.

Now, we need the following variation of Theorem 2.1 in [5].

Lemma 4. *Let H be the union of at most $n-1$ linearly independent complex hyperplanes in $\mathbb{C}^n$, D an open set in $\mathbb{C}^n$, and $K \subset D$ a compact set. Let $\Phi_t : D \to \mathbb{C}^n$, $t \in [0,1]$, be a C^2-smooth isotopy of biholomorphic maps which fix pointwise $D \cap H$ such that $\Phi_t(D \cap H) = \Phi_t(D) \cap H$. Suppose that Φ_0 is the identity map and the set $\Phi_t(K)$ is polynomially convex for every $t \in [0,1]$.*

Then Φ_1 can be approximated, uniformly on K, by automorpisms of $\mathbb{C}^n$, which fix pointwise H.

For a moment, we may assume that Lemma 4 is true. Let $\gamma : [0,1] \to \mathbb{C}^n \setminus (K \cup H)$ be a C^2-smooth path, $\gamma(0) = p$, $\gamma(1) = q$. Then we apply Lemma 4 to the following situation:

take $\Phi_t(z)$ to be z near K and to be $z + \gamma(t) - p$ near p, and choose a sufficiently small neighborhood D of the polynomially convex set $K \cup \{p\}$. For a sufficiently small $\varepsilon > 0$, denote by ψ the corresponding automorphism and set $\tilde{r} := \psi(r)$ for $r \in B$. Let f_1 be the Lagrange interpolation polynomial with

$$f_1(\tilde{r}_2 \dots \tilde{r}_n) = \frac{1}{\tilde{r}_2 \dots \tilde{r}_n} \log \frac{r_1}{\tilde{r}_1}$$

for every $r \in B$. Note that the overshear

$$\psi_1(z) := (z_1 \exp(z_2 \dots z_n f_1(z_2 \dots z_n)), z_2, \dots, z_n)$$

sends $\tilde{r}$ to the point $(r_1, \tilde{r}_2, \dots, \tilde{r}_n)$. It is left to define in a similar way $\psi_2, \dots, \psi_n$ and to consider the composition $\psi_n \circ \dots \circ \psi_1 \circ \psi$. This completes the proof of Lemma 3. □

Proof of Lemma 4. Note that under the assumptions of Lemma 4, there exists a neighborhood $U \subset D$ of K such that $U_t := \Phi_t(U)$ is Runge for each $t \in [0,1]$ (Lemma 2.2 in [5]). We shall follow the proofs of Theorem 1.1 in [5] and Theorem 2.5 in [13]. Consider the vector field

$X_t := \dfrac{d}{dt}\Phi_t \circ \Phi_t^{-1}$ defined on U_t. For a sufficiently large positive integer N and $0 \le j \le N-1$ set

$$X_{j,t} := \begin{cases} 0, & t \notin [j/N,(j+1)/N] \\ X_{j/N}, & t \in [j/N,(j+1)/N]. \end{cases}$$

Note that $X_{j/N}$ vanishes on $U_{j/N} \cap H$. It is easy to see that it can be approximated by holomorphic vector fields on $\mathbb{C}^n$ which vanish on H, since $U_{j/N}$ is Runge (here and below, the approximations are locally uniformly). On the other hand, these vector fields can be approximated by Lie combinations of complete vector fields vanishing on H (Proposition 5.13 in [13]). Thus we may assume that $X_{j/N}$ is a Lie combination of complete vector fields vanishing on H. Note that the local flow of $\sum_{j=0}^{N-1} X_{j,t}$ at time 1 is $h_{N-1} \circ \cdots \circ h_0$, where h_j is the local flow of $X_{j/N}$ at time $\dfrac{1}{N}$. If $N \to \infty$, then this composition converges to the time one map Φ_1 of the flow of X_t. To finish the proof of Lemma 4, it is enough to note that every h_j can be approximated by finite compositions of automorphisms of $\mathbb{C}^n$ which fix H (cf. the proof of Theorem 2.5 in [13]). $\square$

In this way Proposition 2 is completely proved.

Remark. It is an open question whether every holomorphic vector field in $\mathbb{C}^n$, which vanishes on the set $L := \{z_1 \cdots z_n\}$, can be locally uniformly approximated by Lie combinations of complete vector fields vanishing on L [13]. If this would be so, then the above proof shows that Proposition 2 is also true for every union of linearly independent complex hyperplanes in $\mathbb{C}^n$, $n \ge 3$. To see this, choose, for example, the starting embedding

$$\alpha_0(\eta) := (\exp(-\eta^2), \exp(-\eta\sqrt{2}), \exp(\eta), \ldots, \exp(\eta)).$$

It remains an unsolved problem (for us) if there exists a proper holomorphic embedding of $\mathbb{C}$ in $\mathbb{C}^2$ whose graph avoids both coordinate axes.

We are also able to answer the open problem, posed after Proposition 1, in the bounded case.

Proposition 5. *If K is a polynomially convex compact set in $\mathbb{C}^n$, then for any discrete set C of points in $\mathbb{C}^n \setminus K$ there exists a proper holomorphic embedding H of $\mathbb{C}$ in $\mathbb{C}^n$ avoiding K and passing through any of these points. In addition, for a given point $c \in C$ and $X \in$*

$\mathbb{C}^n \setminus \{0\}$ we can choose H such that $H'(H^{-1}(c)) = X$. In particular, the Lempert function and the Kobayashi pseudometric of $\mathbb{C}^n \setminus K$ vanish.

Proof. The proof is a modification of the one of Proposition 2 in [6].

We may assume that $X = (1, 0, \dots, 0)$ and that K does not intersect the first coordinate axis. Note that there exists a smooth nonnegative plurisubharmonic exhaustion function φ on $\mathbb{C}^n$ that is strongly plurisubharmonic on $\mathbb{C}^n \setminus K$ and vanishes precisely on K (cf. [1]). For any $\epsilon > 0$, put

$$G_\varepsilon := \{z \in \mathbb{C}^n : \phi(z) < \varepsilon\} \text{ and } K_\varepsilon := \{z \in \mathbb{C}^n : \phi(z) \le \varepsilon\}.$$

In particular, K_ε is polynomially convex. By Sard's theorem we may choose a strictly decreasing sequence $(\varepsilon_j)_{j \ge 0}$, bounded from below by a positive constant, such that the boundary of $G_j := G_{\varepsilon_j}$ is smooth for any j and $K_0 := K_{\varepsilon_0}$ does not intersect the first coordinate axis. In particular, $K_j := K_{\varepsilon_j}$ has finitely many connected components.

Claim. $K_j \subset \psi_j(K_{j-1})$ for any automorphism ψ_j of $\mathbb{C}^n$ which is closed enough to the identity map on K_{j-1}.

Let now $C = (\alpha_l)_{l \ge 1}$ with $\alpha_1 = c$. Set $H_0(\zeta) = (\zeta, 0, \dots, 0)$ and $\rho_0 = 0$. In view of the claim and the proof of Proposition 2 in [6], for any $j \ge 1$ we may find by induction numbers $\rho_j \ge \rho_{j-1} + 1$, $\zeta_j \in \mathbb{C}$, and an automorphism ψ_j such that for $H_j = \psi_j \circ H_{j-1}$ one has:

(a) $H'_j(\zeta_1) = X$ and $H_j(\zeta_l) = \alpha_l, 1 \le l \le j$;

(b) $|H_j(\zeta)| > |\alpha_j| - 1$ if $|\zeta| \ge \rho_j$ and $K_j \subset \{z \in \mathbb{C}^n : |z| \le |\alpha_j| - \dfrac{1}{2}\}$;

(c) $|H_j(\zeta) - H_{j-1}(\zeta)| \le \delta_j \le 2^{-j}$ if $|\zeta| \le \rho_j$;

(d) $H_j(\mathbb{C}) \cap K_j = \emptyset$.

It is easy to check that the limit map $H := \lim\limits_{j \to \infty} H_j$ exists and that it has the required properties except properness. The last one can be provided by the choice of δ_j. Note that the only modifications that have to be made in the proof of Proposition 2 in [6] are the choice of the ψ_j with the additional property $\psi'_j(\zeta_1)$ to be the unitary matrix and the replacing of the set $F := \{z \in \mathbb{C}^n : |z| \le |\alpha_j| - \dfrac{1}{2}\} \cup H_{j-1}\{|\zeta| \le \rho\}$ by the set $F := K_j \cup H_{j-1}\{|\zeta| \le \rho\}$ if $K_j \not\subset \{z \in \mathbb{C}^n : |z| \le |\alpha_j| - \dfrac{1}{2}\}$.

Proof of the claim. Since K_j has finitely many connected components $K_{j,1}, \dots, K_{j,m}$, we have that $\operatorname{dist}(K_j, \partial K_{j-1}) > 0$. Then we find an $r > 0$ with $\operatorname{dist}(K_j, \partial K_{j-1}) > r$ and some ball B_l with radius r belonging to $K_{j,l}$, $1 \le l \le m$. It follows that $K_j \subset \psi_j(K_{j-1})$, if

$$\max\{|\psi_j(z) - z| : z \in K_{j-1}\} \le r.$$

Indeed, suppose the contrary, i.e., $\psi_j(a) \in K_j$ for some $a \notin K_{j-1}$. We may assume that $\psi_j(a) \in K_{j,1}$. Denote by b_1 the image of the center of B_1 under ψ_j. Then there exists a path γ in $K_{j,1}$ joining $\psi_j(a)$ and b_1. Note that $\psi_j^{-1}(\gamma) \cap \partial K_{j-1} \neq \emptyset$. If $c \in \psi_j^{-1}(\gamma) \cap \partial K_{j-1}$, then $\psi(c) \in K_j$. Hence $r \geq |\psi_j(c) - c| \geq \mathrm{dist}(K_j, \partial K_{j-1})$; a contradiction. □

Note that if F is a proper subset in $\mathbb{C}^2$ such that for any point in $\mathbb{C}^2 \setminus F$ there exists a non-constant entire curve $\gamma : \mathbb{C} \to \mathbb{C}^2 \setminus F$ which passes through this point, then the interior of F is pseudoconvex, since $\mathbb{C}^2 \setminus \overline{\gamma(\mathbb{C})}$ is pseudoconvex [12]. Moreover, if F is compact and for any point $a \in \mathbb{C}^2 \setminus F$ there exists a proper holomorphic mapping $\varphi : \mathbb{C} \to \mathbb{C}^2$ with $a \in \varphi(\mathbb{C}) \subset \mathbb{C}^2 \setminus F$, then F is rational convex [3]. The same does not holds in higher dimensions. For example, if F and G are two proper closed subsets of $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively, then for any point in $\mathbb{C}^{k+l} \setminus (F \times G)$ there exists a proper holomorphic embedding of $\mathbb{C}$ in $\mathbb{C}^{k+l}$ avoiding $F \times G$ and passing through this point.

The next proposition is in the spirit of the above remark and it generalizes Proposition 1 in [8].

Proposition 6. *If F and G are two sets in $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively, then for any countably set C of points in $\mathbb{C}^{k+l}$ with $\mathrm{dist}(C, F \times G) > 0$ there exists a holomorphic immersion of $\mathbb{C}$ in $\mathbb{C}^{k+l}$ avoiding $F \times G$ and passing through any point of C.*

Proof. The idea for the proof comes from the one of Theorem 2 in [10]. For any point c in $\mathbb{C}^{k+l}$ denote by c' and c'' its projections onto $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively. Set $\varepsilon := \mathrm{dist}(C, F \times G) > 0$, $C' := \{c \in C : \mathrm{dist}(c', \mathbb{C}^k \setminus F) \geq \varepsilon\}$ and $C'' := C \setminus C'$. We may assume that both sets are infinite and enumerate them, i.e. $C' = (a_j)_{j \geq 0}$ and $C'' = (b^j)_{j \geq 0}$. Denote by $\mathbb{D}_n(c, r)$ the polydisc in $\mathbb{C}^n$ with center at c and radius r. Note that $\mathbb{D}_k(a_j', \varepsilon) \subset \mathbb{C}^k \setminus F$ and $\mathbb{D}_l(b_j'', \varepsilon) \subset \mathbb{C}^l \setminus G$ for any $j \geq 0$. Define

$$A_j := \{z \in \mathbb{C} : \mathrm{Re}(z) \leq -3, |\,\mathrm{Im}(z) - 7j| \leq 3\},\ j \geq 1,$$

$$A_0 := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq -1\} \setminus \cup_{j=1}^{\infty} \{z \in \mathbb{C} : \mathrm{Re}(z) > 5, |\,\mathrm{Im}(z) - 7j| < 1\}.$$

Choose a number $t \in (0, 1)$ such that

$$t \exp(\sqrt[3]{x} - \sqrt[4]{x+2}) \geq 4e(1-t)\sqrt[3]{(x+2)^4},\ x \geq 0.$$

For $1 \leq m \leq k$, combining the extensions of Arakelian's theorem in [4, 10] gives an entire function f_m such that

$$|f_m(z) - a_{0,m} - \frac{\varepsilon}{2} t \exp(-\sqrt[4]{z+2})| < \frac{\varepsilon}{2}(1-t)\exp(-\sqrt[3]{|z|}), z \in A_0,$$

$$|f_m(z) - a_{j,m} - \frac{\varepsilon}{2} t \exp(-\sqrt[4]{-z-2})| < \frac{\varepsilon}{2}(1-t)\exp(-\sqrt[3]{|z|}), z \in A_j, j \geq 1,$$

$f_m(2) = a_{0,m},\ f_m(-2) = b_{0,m},\ f_m(-7+i7j) = a_{j,m},\ f_m(7+i7j) = b_{j,m}$ for $j \geq 1$ ($\sqrt[4]{z}$ is the branch with $\sqrt[4]{1} = 1$ and $c_{j,m}$ denotes the m-th coordinate of the point c_j). Note that $|f_m(z) - a_{j,m}| < \varepsilon$ if $z \in A_j$. For $k+1 \leq m \leq k+l$ we choose analogously an entire function f_m such that

$$|f_m(z) - b_{0,m} - \frac{\varepsilon}{2} t \exp(-\sqrt[4]{-z-2})| < \frac{\varepsilon}{2}(1-t)\exp(-\sqrt[3]{|z|}), -z \in A_0,$$

$$|f_m(z) - b_{j,m} - \frac{\varepsilon}{2} t \exp(-\sqrt[4]{z+2})| < \frac{\varepsilon}{2}(1-t)\exp(-\sqrt[3]{|z|}), -z \in A_j, j \geq 1,$$

$f_m(2) = a_{0,m},\ f_m(-2) = b_{0,m},\ f_m(-7+i7j) = a_{j,m},\ f_m(7+i7j) = b_{j,m}$ for $j \geq 1$. Then the mapping $(f_1, \dots, f_{k+l})$ will have the required properties if it is non-singular. To see this, note that applying the triangle inequality and the Cauchy inequality gives

$$\Big|\frac{\varepsilon t}{8\sqrt[3]{|z+2|^4}} \exp(-\sqrt[4]{|z+2|})\Big| - |f'_m(z)| < \frac{\varepsilon}{2}(1-t)\exp(1 - \sqrt[3]{|z|})$$

for $1 \leq m \leq k$ and

$$z \in E_0 := \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\} \setminus \bigcup_{j=1}^{\infty} \{z \in \mathbb{C} : \operatorname{Re}(z) > 4, |\operatorname{Im}(z) - 7j| < 2\}.$$

Then the choice of t shows that $f'_m(z) \neq 0$ if $1 \leq m \leq k$ and $z \in E_0$; a similar argument gives that $f'_m(z) \neq 0$ if

$$z \in E := \bigcup_{j=1}^{\infty} \{z \in \mathbb{C} : \operatorname{Re}(z) \leq -4, |\operatorname{Im}(z) - 7j| \leq 2\}.$$

We obtain analogously that $f'_m(z) \neq 0$ if $k+1 \leq m \leq k+l$ and $-z \in E_0 \cup E$, which implies that the mapping is non-singular. $\square$

Note that, in general, the mapping in Proposition 6 cannot be chosen to be proper. For example, let $F := \mathbb{C} \setminus \mathbb{D}_1(0,1)$ and let $f := (f_1, f_2)$ be a proper holomorphic map of $\mathbb{C}$ in $\mathbb{C}^2$ which avoids $F \times F$. Choose an R such that $\max\{|f_1(z)|, |f_2(z)|\} \geq 2$ for $|z| > R$. Assume that f_1 is not a polynomial. Then by Picard's theorem there is a point $a \in \mathbb{C}$, $|a| > R$, with $|f_1(a)| = 1$. Thus $|f_2(a)| \geq 2$. On the other side, using that $f(\mathbb{C}) \cap (F \times F) = \emptyset$ implies that $|f_1(a)| < 1$, a contradiction. In conclusion, one of the functions f_1 and f_2 is a polynomial and the other one is a constant smaller than 1.

It follows from Proposition 6 that if F and G are two closed proper subsets of $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively, then the Lempert function of $\mathbb{C}^{k+l} \setminus (F \times G)$ vanishes. The next proposition implies that the same holds for the Kobayashi pseudometric.

Proposition 7. *If F and G are two proper closed sets in $\mathbb{C}^k$ and $\mathbb{C}^l$, respectively, then for any point $c \in \mathbb{C}^{k+l} \setminus (F \times G)$ and any vector $X \in \mathbb{C}^{k+l}$ there exists a holomorphic mapping of $\mathbb{C}$ in $\mathbb{C}^{k+l} \setminus (F \times G)$ with $f(0) = c$ and $f'(0) = X$.*

Proof. We may assume that $c' \in \mathbb{C}^k \setminus F$ and $\mathbb{D}_l(0,1) \subset \mathbb{C}^l \setminus G$. The statement is trivial if $X' = 0$. Otherwise, we may assume $c' = 0$ and the ball in $\mathbb{C}^k$ with center at the origin and radius $(e+1)\sqrt{k}$ belongs to $\mathbb{C}^k \setminus F$. After a unitary transformation of $\mathbb{C}^k$ we may also assume that $X' = (r, \dots, r)$ for some $r > 0$. Note that $\mathbb{D}_k(0, e+1) \subset \mathbb{C}^k \setminus F$ and if $|e^{rz} - 1| \geq e+1$, then $\mathrm{Re}(z) \geq \dfrac{1}{r}$. By Arakelian's theorem, there exists an entire function f_m such that $f_m(0) = 0$, $f'_m(0) = X_m$, and $|f_m(z)| < 1$ if $\mathrm{Re}(z) \geq \dfrac{1}{r}$, $k+1 \leq m \leq k+l$. Setting $f_m(z) := e^{rz} - 1$ for $1 \leq m \leq k$ implies that the mapping $(f_1, \dots, f_{k+l})$ has the required properties. □

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